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## LETTER TO THE EDITOR

## Spectral singularities of non-Hermitian Hamiltonians and SUSY transformations

**Boris F Samsonov**

Department of Physics, Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

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### Abstract

Simple examples of non-Hermitian Hamiltonians with purely real spectra defined in  $L^2(\mathbb{R}^+)$  having spectral singularities inside the continuous spectrum are given. It is shown that such Hamiltonians may appear by shifting the independent variable of a real potential into the complex plane. Also they may be created as SUSY partners of Hermitian Hamiltonians. In the latter case spectral singularities of a non-Hermitian Hamiltonian are ordinary points of the continuous spectrum for its Hermitian SUSY partner. Conditions for transformation functions are formulated when a complex potential with complex eigenenergies and spectral singularities has a SUSY partner with a real spectrum without spectral singularities. Finally, we shortly discuss why Hamiltonians with spectral singularities are ‘bad’.

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There are two essential differences between non-Hermitian Hamiltonians having a purely real spectrum and Hermitian Hamiltonians. The first difference is related to the discrete spectrum and consists in the possibility of the appearance of *associated functions* [1, 2] (also called in [3] ‘background eigenfunctions’), which are not eigenfunctions of the Hamiltonian but they should be added to the set of discrete spectrum eigenfunctions to complete a basis in the corresponding Hilbert space. Hamiltonians of this kind are known as non-diagonalizable. In my previous letter [3] it was shown that they can be transformed into diagonalizable forms by appropriate *SUSY transformations*. In this letter I show that SUSY transformations may be useful to ‘cure’ another ‘disease’ of non-Hermitian Hamiltonians which is related just to the second difference between Hermitian and non-Hermitian Hamiltonians consisting of the appearance of *spectral singularities* inside a continuous spectrum.

The paper by Naimark [4] was one of the first most essential contributions to the spectral theory of a non-self-adjoint operator of Schrödinger type. In particular, he was the first who noted the possibility of the appearance of spectral singularities inside a continuous

spectrum. Later Lyantse [5] (see also [2]) studied carefully some properties of Hamiltonians with spectral singularities. In this letter using the simplest real nontrivial potential without a discrete spectrum we demonstrate that the usual practice of getting exactly solvable complex potentials with a purely real spectrum consisting of simple shifting of the dependent variable of a real potential to the complex plane may lead to a potential with spectral singularities. Then we show how a spectral singularity may appear after a SUSY transformation over a real potential and how it can be ‘removed’ (more precisely it can be transformed into an ordinary point of the continuous spectrum of a SUSY partner Hamiltonian). Finally we briefly discuss why Hamiltonians possessing spectral singularities are ‘bad’.

For simplicity we will consider Sturm–Liouville problems on the positive semiaxis since in this case the continuous spectrum is non-degenerate and spectral singularities are easier ‘to handle’. Nevertheless, we would like to point out that interested reader can find a deep study of non-Hermitian Hamiltonians defined on the whole real line, which includes the strict formulation and proof of the inverse scattering theorem, in [6]. (We note that spectral singularities constitute a part of the spectral data.) We start with the definition of spectral singularities of a non-Hermitian one-dimensional Hamiltonian mainly following the paper by Lyantse [5].

Let us have a complex-valued function  $V(x)$  such that

$$\int_0^{\infty} e^{\varepsilon x} |V(x)| dx < \infty \quad (1)$$

where  $\varepsilon$  is a positive number. We call such functions *exponentially decreasing* at infinity. We note that this condition simplifies essentially studying non-Hermitian Hamiltonians with spectral singularities although they may appear under a weaker condition that  $V(x)$  is absolutely summable on the positive semiaxis. In particular, condition (1) guarantees a finite number of spectral singularities.

Consider the differential expression

$$l[y] = -y'' + V(x)y \quad (2)$$

and the boundary condition

$$y(0) = 0. \quad (3)$$

Let  $h$  be the (non-self-adjoint) operator created by differential expression (2) and condition (3) in the Hilbert space  $L^2(R^+)$  where  $R^+$  is the positive semiaxis  $R^+ = (0, \infty)$ . The domain of definition of  $h$ ,  $\mathcal{D}(h)$ , consists of all functions  $\psi$ , which have absolutely continuous derivative in any finite interval  $(\alpha, \beta) \subset R^+$ , satisfy the boundary condition (3) and are such that both  $\psi \in L^2(R^+)$  and  $l[\psi] \in L^2(R^+)$ . If  $\psi \in \mathcal{D}(h)$  then by definition  $h\psi = l[\psi]$ , so we also denote  $h = -\partial_x^2 + V(x)$  and call  $h$  Hamiltonian.

For any potential  $V(x)$  satisfying condition (1) there exists a solution  $e_+(x, s)$  (Jost solution) of the equation  $l[y] = s^2 y$  with the following asymptotical behaviour at  $x \rightarrow \infty$ :

$$\frac{d^j}{ds^j} e_+(x, s) = (ix)^j e^{isx} + o(e^{-\frac{\varepsilon}{2}x}), \quad j = 0, 1, \dots \quad (4)$$

which is uniform with respect to  $s$  in any half-plane  $\text{Im } s \geq -\eta$  where  $0 < \eta < \frac{\varepsilon}{2}$ . Put

$$A(s) = e_+(0, s). \quad (5)$$

The function  $A(s)$  (Jost function) is holomorphic in the half-plane  $\text{Im } s > -\frac{\varepsilon}{2}$  and its asymptotics

$$A(s) = is[1 + o(1)] \quad |s| \rightarrow \infty \quad (6)$$

implies that the equation  $A(s) = 0$  has a finite number of solutions in the half-plane  $\text{Im } s \geq 0$ . Let  $s_1, \dots, s_r$  be solutions to the equation  $A(s) = 0$  in the open upper half-plane  $\text{Im } s > 0$ . Then the spectrum of the operator  $h$  has the discrete part defined as  $E_1 = s_1^2, \dots, E_r = s_r^2$  and the continuous part filling the non-negative semiaxis  $E \geq 0$ .

For a self-adjoint Hamiltonian the Jost function has no real zeros [8]. But for a non-self-adjoint case it may happen that the equation  $A(s) = 0$  has real roots. Let  $\sigma_1, \dots, \sigma_\rho$  be real non-zero solutions to this equation. We would like to stress that they may be both positive and negative. Since  $\sigma_1^2 > 0, \dots, \sigma_\rho^2 > 0$ , the points

$$\tilde{E}_1 = \sigma_1^2, \dots, \tilde{E}_\rho = \sigma_\rho^2 \tag{7}$$

belong to the continuous spectrum of  $h$ .

**Definition 1.** Points (7) are called spectral singularities of the operator  $h$ .

**Remark 1.** We would like to stress the difference between spectral singularities and other exceptional points that may appear inside the continuous spectrum which are known as bound states embedded into the continuum (BEICs, see, e.g., [7] and references therein). In contrast with spectral singularities, which belong to the continuous spectrum, BEICs correspond to the discrete spectrum of a Hamiltonian.

Inside the strip  $|\text{Im } s| \leq \frac{\varepsilon}{2}$  the functions  $e_+(x, s)$  and  $\tilde{e}_-(x, s) = e_+(x, -s)$  form a fundamental set of solutions to the equation  $l[y] = s^2 y$  and their Wronskian is  $W(e_+, \tilde{e}_-) := e_+(x, s)\tilde{e}'_-(x, s) - e'_+(x, s)\tilde{e}_-(x, s) = -2is$  where the prime denotes the derivative with respect to  $x$ . From here it follows that the function defined as

$$\psi(x, s^2) = \frac{A(-s)e_+(x, s) - A(s)e_+(x, -s)}{-2is} \quad |\text{Im } s| \leq \frac{\varepsilon}{2} \tag{8}$$

satisfies the condition  $\psi(0, s^2) = 0$  and  $\psi'(0, s^2) = 1$ .

For any  $s$  from the half-plane  $\text{Im } s > 0$  there exists a function  $e_-(x, s)$  exponentially growing as  $x \rightarrow \infty$  which in this case is a solution linearly independent of  $e_+(x, s)$  and the Wronskian of  $e_+$  and  $e_-$  has the same value as in the previous case,  $W(e_+, e_-) = -2is$ . Therefore we can also write

$$\psi(x, s^2) = \frac{B(s)e_+(x, s) - A(s)e_-(x, s)}{-2is} \quad \text{Im } s > 0 \tag{9}$$

where  $B(s) = e_-(0, s)$ . Keeping in mind the case of a self-adjoint operator  $h$  we can say that at  $s^2 = E_1, \dots, s^2 = E_r$ ,  $\psi(x, s^2)$  describes (unnormalized) bound states of the Hamiltonian  $h$  and for  $\text{Im } s = 0$  this is a scattering state. We would like to stress that the physical contents of a scattering of a particle on a complex potential need a special analysis which is beyond the scope of the current letter and will be discussed in a separate publication.

Let us consider the well-known exactly solvable potential

$$V(x) = \frac{-2a^2}{\cosh^2(ax)} \quad a > 0. \tag{10}$$

Being considered on the whole real line this is a transparent one-level (one-soliton) potential but on the positive semiaxis it has only a continuous spectrum. Let us shift the space variable to the complex plane and consider

$$V(x) = \frac{-2a^2}{\cosh^2(ax + b)} \quad 0 \leq x < \infty \quad b \in \mathbb{C} \quad a > 0. \tag{11}$$

This potential evidently satisfies the condition (1) for any  $\varepsilon < 2a$ .

It is not difficult to find the Jost solution

$$e_+(x, k) = \frac{k + ia \tanh(b + ax)}{k + ia} e^{ikx} \quad (12)$$

which according to (5) gives the Jost function

$$A(k) = \frac{k + ia \tanh(b)}{k + ia}. \quad (13)$$

Here and in what follows we are using the notation  $k = s$  for  $\text{Im } s = 0$ . The function (13) vanishes at  $k = k_0 = -ia \tanh b$  which is real provided  $b$  is purely imaginary. Thus, for a purely imaginary  $b$  potential (10) at  $k = k_0 = -ia \tanh b$  has a spectral singularity  $E = k_0^2$ . The position of this point does not depend on the sign of  $k_0$  and in particular on the sign of the parameter  $b$  but the shape of the potential (11) changes with the change of this sign.

Now we are going to show how similar Hamiltonians occur after SUSY transformations over Hermitian Hamiltonians. Since such transformations can always be realized in the opposite direction this opens the possibility of converting (non-Hermitian) Hamiltonians with spectral singularities into Hermitian Hamiltonians.

Let us denote  $V = V_0$  in (2). We will suppose that  $V_0(x)$  is a real scattering potential (see, e.g., [8]), i.e. it satisfies the condition

$$\int_0^\infty x |V_0(x)| dx < \infty. \quad (14)$$

In this case the operator  $h = h_0 = -\partial_x^2 + V_0(x)$  is essentially self-adjoint in the Hilbert space  $L^2(\mathbb{R}^+)$  and has a real and simple spectrum which has a finite number (which may be equal to zero) of discrete levels and a continuous part filling the non-negative semiaxis [8]. We will denote by  $\psi_n(x)$  its discrete spectrum eigenfunctions and by  $\psi_k(x)$ ,  $E = k^2 > 0$  the continuous spectrum eigenfunctions keeping the notation  $\psi_E(x)$  for a solution to the differential equation  $-\psi_E''(x) + [V_0(x) - E]\psi_E(x) = 0$ .

To simplify notation in what follows we will use symbols  $h_0, h_1, h_2$  to define both the operator-differential expressions  $h_j = -\partial_x^2 + V_j(x)$ ,  $j = 0, 1, 2$  and corresponding operators in the Hilbert space  $L^2(\mathbb{R}^+)$  calling them Hamiltonians. We hope this will not cause trouble for the reader.

To get a complex potential  $V_2(x)$  defining the Hamiltonian  $h_2 = -\partial_x^2 + V_2(x)$ , which is a 2-SUSY partner to the Hermitian  $h_0$ , we are using *second-order SUSY transformations* considered in the necessary detail in [3]. Below we formulate conditions which should be imposed on transformation functions leading to the appearance of a spectral singularity in the spectrum of  $h_2$ . Accordingly, the opposite transformation removes the spectral singularity and gives back (Hermitian) Hamiltonian  $h_0$ . If the 'wave function' (we are using physical terminology for non-Hermitian Hamiltonians also) corresponding to the spectral singularity is nodeless it can be used for removing the spectral singularity by a simpler first-order SUSY transformation.

Before formulating the main result of the present letter we recall very briefly the necessary formulae defining 1- and 2-SUSY transformations.

If the first-order transformation is implemented over the potential  $V_0$  the 1-SUSY partner potential  $V_1(x)$  is given by

$$V_1(x) = V_0(x) - 2w'(x) \quad w(x) = [\log u(x)]' \quad (15)$$

where  $u(x)$  is a nodeless  $\forall x \in \mathbb{R}^+$  solution to the equation  $h_0 u = \alpha u$ . It is called the *transformation function*. It can vanish at the origin thus giving a potential singular at the origin. The solutions to the transformed equation  $h_1 \varphi_E = E \varphi_E$ ,  $h_1 = -\partial_x^2 + V_1(x)$  are expressed in terms of solutions  $\psi_E(x)$  to the initial equation

$$\varphi_E(x) = -\psi_E'(x) + w(x)\psi_E(x) \quad E \neq \alpha \quad (16)$$

$$\varphi_\alpha(x) = \frac{1}{u(x)}. \quad (17)$$

Choice  $\varphi_\alpha = \frac{1}{u}$  as the transformation function for the next 1-SUSY transformation corresponds to the backward transformation from  $h_1$  to  $h_0$ .

In the case of a second-order transformation over the potential  $V_0$  the 2-SUSY partner potential  $V_2$  reads

$$V_2 = V_0 - 2[\log W(u_1, u_2)]'' \quad (18)$$

and the solutions to the equation  $h_2\varphi_E = E\varphi_E$ ,  $h_2 = -\partial_x^2 + V_2(x)$  are defined as follows:

$$\varphi_E = (E - \alpha_2)\psi_E + (\alpha_1 - \alpha_2)\frac{W(u_2, \psi_E)}{W(u_1, u_2)}u_1 \quad (19)$$

$$= (E - \alpha_1)\psi_E + (\alpha_1 - \alpha_2)\frac{W(u_1, \psi_E)}{W(u_1, u_2)}u_2 \quad E \neq \alpha_1, \alpha_2 \quad (20)$$

$$\varphi_{\alpha_{1,2}} = \frac{u_{2,1}}{W(u_1, u_2)}. \quad (21)$$

Here  $u_1, u_2$  (they are called *transformation functions*) and  $\psi_E$  are solutions to the equation  $h_0 y = E y$  corresponding to the eigenvalues  $\alpha_1, \alpha_2$  ( $\alpha_1 \neq \alpha_2$ ) and  $E$  respectively; the symbol  $W(\cdot, \cdot)$  is reserved for the Wronskian. If we realize the same transformation once again choosing  $h_2$  as the initial Hamiltonian and the functions (21) as the transformation functions we will go back to the potential  $V_0$ . Thus, the procedure is completely reversible.

Now we are able to prove the main result of the present letter.

**Theorem 1.** *Let a real-valued function  $V_0(x)$ ,  $x \in \mathbb{R}^+$  be smooth, finite at  $x = 0$  and exponentially decreasing at infinity. Then there exists a 2-SUSY transformation such that a complex potential  $V_2(x)$  obtained with the help of formula (18) is exponentially decreasing and regular  $\forall x \in [0, \infty)$ ; the Hamiltonian  $h_2 = -\partial_x^2 + V_2(x)$  has a spectral singularity at  $E = k_0^2$  where  $k_0$  is an arbitrary non-zero real number. There exists a real 1-SUSY partner  $V_1(x)$  of the potential  $V_2(x)$  singular only at the origin with the singularity strength equal to 1. The Hamiltonian  $h_1 = -\partial_x^2 + V_1(x)$  is essentially self-adjoint in  $L^2(\mathbb{R}^+)$ .*

**Remark 2.** We recall (see, e.g., [9]) that the singularity strength  $\nu$  of a potential  $V(x)$  is defined by its behaviour at the origin:

$$V(x) \rightarrow \frac{\nu(\nu+1)}{x^2} \quad x \rightarrow 0. \quad (22)$$

**Remark 3.** Our proof of the theorem is constructive so that below we give a precise recipe how such potentials may be obtained.

**Proof.** To prove the first part of the statement we formulate the conditions which should be imposed on transformation functions  $u_1$  and  $u_2$  leading according to (18) to the potential  $V_2$  with desirable properties. First we note that  $V_0(x)$  satisfies condition (14) and any solution to the equation  $(h_0 - E)\psi_E = 0$  with the given  $E \in \mathbb{C}$  at the origin either vanishes (regular solution) or takes a finite non-zero value (irregular solution). It follows from here and (19), (20) that to preserve the zero boundary condition at the origin the method should involve transformation functions  $u_1(x)$  and  $u_2(x)$  such that at least one of them vanishes at the origin.

Let  $u_1(x)$  be real and  $u_1(0) = 0$ ,  $\alpha_1 < 0$ . Choose  $u_2(x)$  coinciding with the Jost solution  $u_2(x) = e_+(x, k_0)$  ( $\alpha_2 = k_0^2 > 0$ ). Formula (18) gives a regular potential  $V_2(x)$  provided the Wronskian of  $u_1$  and  $u_2$  is nodeless in  $\mathbb{R}^+$ . Therefore first of all we have to convince ourselves that this property of the Wronskian takes place for the given  $u_1$  and  $u_2$ .

According to the Sturm oscillator theorem (see, e.g., [10])  $u_1(x)$  has no nodes in  $R^+$ . The same property exists for  $u_2(x)$  also. Indeed, since  $u_2(x)$  is a complex solution to the Schrödinger equation with a real potential  $V_0(x)$  it can always be presented as  $u_2(x) = y_1(x) + iy_2(x)$  where real-valued functions  $y_1(x)$  and  $y_2(x)$  are two linearly independent solutions to the same equation with the same value of  $E = \alpha_2$ . Since they are real and correspond to  $E > 0$  they have an oscillating character but their nodes never coincide. Otherwise their Wronskian which is  $x$ -independent would vanish, which is impossible.

We shall now show that  $u_1$  and  $u_2$  have a non-vanishing Wronskian  $\forall x \in [0, \infty)$ . Denote for brevity  $W = W(u_1, u_2)$  which is a smooth complex-valued function of the real variable  $x$ . Let  $f$  be its modulus,  $f = \sqrt{WW^*}$ , where the asterisk denotes the quantity complex conjugate to the given one and only the positive branch of the square root should be used. Suppose that the function  $W = W(x)$  vanishes at any  $x = x_0 \in R^+$ ,  $W(x_0) = 0$ . Since  $f(x) \geq 0 \forall x \in R^+$  the point  $x = x_0$  is the point of a minimum for  $f(x)$ . Then since  $W^*(x_0) = 0$ , the ratio  $W^*/W$  is undetermined. Using the l'Hospital rule we find that  $W^*/W = (W^*)'/W'$  at  $x = x_0$ . From here we get

$$f'(x_0) = \frac{W'}{2} \sqrt{\frac{W^*}{W}} + \frac{(W^*)'}{2} \sqrt{\frac{W}{W^*}} = |W'(x_0)|.$$

Using the Schrödinger equation and already established fact that both  $u_1(x)$  and  $u_2(x)$  are nodeless in  $R^+$  we can easily see that  $W'(x) = (\alpha_2 - \alpha_1)u_1(x)u_2(x)$  does not vanish  $\forall x \in R^+$ . This result means that  $f'(x_0) \neq 0$  which contradicts the fact that  $x_0$  is the point of a minimum for  $f(x)$ . This contradiction proves the nodeless character of  $W(x) \forall x \in R^+$ . Therefore taking into consideration that  $W(0) = -u_1'(0)u_2(0) \neq 0$  we see that the potential  $V_2(x)$  given in (18) is regular  $\forall x \in [0, \infty)$ . The fact that  $V_2(x)$  is exponentially decreasing follows from formula (18) and the asymptotical behaviour of the transformation functions  $u_1(x)$  and  $u_2(x)$ .

Now we shall prove that the point  $E = k_0^2$  of the continuous spectrum of the Hamiltonian  $h_2 = -\partial_x^2 + V_2(x)$  with  $V_2(x)$  given in (18) is a spectral singularity. Indeed, according to (21) the condition  $\varphi_{\alpha_2}(0) = 0$  follows from the regular character of  $u_1(x)$ . Necessary asymptotical behaviour of  $\varphi_{\alpha_2}(x)$  follows from the asymptotics of the Jost solution  $e(x, k_0)$  and the fact that the logarithmic derivative of  $u_1(x)$  is asymptotically constant. Thus, this is just the point  $E = \alpha_2 = k_0^2$  which is a spectral singularity for  $h_2$ . We note that the choice  $u_2(x) = e(x, -k_0)$  produces another potential with the same spectral singularity at  $E = k_0^2$ .

To prove the last statement of the theorem we note that the second-order transformation leading to  $V_2$  from the given  $V_0$  can always be presented as a chain of two transformations  $V_0 \rightarrow V_1 \rightarrow V_2$  where a real-valued potential  $V_1$  is obtained from  $V_0$  by the first-order transformation (15) with a real transformation function  $u = u_1(x)$  vanishing only at the origin and this zero is simple. Therefore  $V_1(x)$  has only one singular point (poles)  $x = 0$  and its behaviour near  $x = 0$  is given by (22). The regular potential  $V_2$  is obtained from  $V_1$  by another first-order transformation with the transformation function  $\tilde{u}(x) = -u_2'(x) + w(x)u_2(x)$  where  $w(x) = u_1'(x)/u_1(x)$ . Therefore using the transformation function  $1/\tilde{u}(x)$  which is just the continuous spectrum eigenfunction corresponding to the spectral singularity  $E = k_0^2$  we go back from  $V_2$  to  $V_1$  transforming in this way the complex potential  $V_2$  having a spectral singularity at  $E = k_0^2$  into the real potential  $V_1$  without spectral singularities but singular at  $x = 0$  with the singularity strength equal to 1. The value 1 for the singularity strength follows from the fact that the zero at  $x = 0$  of the transformation function  $u_1(x)$  is simple. The Hamiltonian  $h_1$  is essentially self-adjoint since the potential  $V_1(x)$  is real, finite in  $R^+$  and exponentially decreasing at infinity (the 1-SUSY transformation also does not change the asymptotical behaviour of the potential).  $\square$

From the last lines of the proof we can draw the following deductions.

**Corollary 1.** *Suppose that the continuous spectrum eigenfunction  $\psi_{k_0}(x)$  of a regular in  $R^+$  complex potential  $V_2(x)$ , which is a 2-SUSY partner of a real and regular  $V_0(x)$ , corresponds to a spectral singularity  $E = k_0^2$  such that the zero  $k = k_0$  of the Jost function  $A(k)$  is simple, then  $\psi_{k_0}(x)$  is nodeless.*

More important for us is the following implication.

**Corollary 2.** *Let a complex potential  $V_2(x)$  have only one spectral singularity  $E = k_0^2$ , the zero  $k = k_0$  of the Jost function  $A(k)$  is simple and continuous spectrum eigenfunction  $\psi_{k_0}(x)$  is nodeless in  $R^+$ . Then  $V_2(x)$ , which is 1-SUSY partner obtained with the help of (15) with  $u = \psi_{k_0}$ , has no spectral singularities and is singular only at the origin with the singularity strength equal to 1.*

For instance, for  $u = \varphi_{k_0}$  where  $\varphi_{k_0}$  is given in (12) at  $k = k_0 = -ia \tanh b$ , the potential  $V_1$  which is 1-SUSY partner of  $V_2$  given in (11) reads

$$V_1 = \frac{2a^2}{\sinh^2(ax)}.$$

The last statement while combined with the results of the paper [3] leads to the following theorem which we leave without proving.

**Theorem 2.** *Let finite at the origin and exponentially decreasing complex potential  $V_{N+n}(x)$  be such that the equation  $A(k) = 0$ , where  $A(k)$  is the Jost function, has only simple roots, between which  $n$  non-zero roots  $k_0, \dots, k_{n-1}$  are real (they correspond to spectral singularities  $E_j = k_j^2, j = 0, \dots, n - 1$  of the Hamiltonian  $h_{N+n} = -\partial_x^2 + V_{N+n}$ ) and  $N$  roots  $s_1, \dots, s_N$  lay in the open upper half of the complex  $s$ -plane outside the imaginary axis ( $E_l = s_l^2, l = 1, \dots, N$  are points of the discrete spectrum of  $h_{N+n}$ ). If the Wronskian  $W(\tilde{\psi}_{s_1}, \dots, \tilde{\psi}_{s_N}, \psi_{k_0}, \dots, \psi_{k_{n-1}})$ , where  $\tilde{\psi}_{s_l}$  are discrete spectrum eigenfunctions and  $\psi_{k_j}, j = 0, \dots, n - 1$  are continuous spectrum eigenfunctions, does not vanish in  $R^+$  then the potential  $V_{N+n}(x)$  is a  $(N + n)$ -SUSY partner of the potential*

$$V_0(x) = V_{N+n} - 2[\log W(\psi_{s_1}, \dots, \psi_{s_N}, \psi_{k_0}, \dots, \psi_{k_{n-1}})]'' \tag{23}$$

which has a real spectrum, has no spectral singularities and is singular at the origin with the singularity strength equal to  $N + n$ .

Let us consider a few examples illustrating theorem 1.

*Example 1.* Take  $V_0 = 0, u_1(x) = \sinh(a_1x) (\alpha_1 = -a_1^2), u_2 = \exp(-ik_0x)$ . Formula (18) gives the potential

$$V_2(x) = -\frac{2a_1^2(a_1^2 + k_0^2)}{[a_1 \cosh(a_1x) - ik_0 \sinh(a_1x)]^2} \tag{24}$$

which at  $a_1 = a$  and  $k_0 = ia \tanh b$  is easily identified as the one given in (11). This means that the potential (11) having a spectral singularity is 2-SUSY partner for the zero potential having no spectral singularities.

*Example 2.* Take  $V_0 = -6 \operatorname{sech}^2 x$ . This is a one-level potential with the ground state function  $\psi_0 = \sqrt{3} \tanh x \operatorname{sech} x (E_0 = -1)$  and the Jost solution

$$e(x, k) = \frac{e^{ikx}}{(k + i)(k + 2i)}(k^2 - 2 + 3 \operatorname{sech}^2 x + 3ik \tanh x). \tag{25}$$

The choice  $u_1 = \psi_0$  and  $u_2 = e(x, k_0)$  leads to the elimination of the ground state level from the spectrum of  $V_1$  and gives the potential (24) at  $a_1 = 2$ . Since SUSY transformations are



reversible this result means that the potential (24) at  $a_1 = 2$  is SUSY partner for both  $V_0 = 0$  and  $V_0 = -6 \operatorname{sech}^2 x$ .

*Example 3.* Take finally  $V_0 = -20 \operatorname{sech}^2 x$ . The Hamiltonian  $h_0$  has two discrete levels:  $E_0 = -9$ ,  $\psi_0 = \sqrt{105/8} \operatorname{sech}^3 x \tanh x$  and  $E_1 = -1$ ,  $\psi_1 = \sqrt{5/8} [2 \cosh(2x) - 5] \operatorname{sech}^3 x \tanh x$ . The Jost solution has the form

$$e(x, k) = \frac{e^{ikx}}{(k+i)(k+2i)(k+3i)(k+4i)} [k^4 - 35k^2 + 24 + 105 \operatorname{sech}^4 x + 10ik(k^2 - 5) \tanh x + 15 \operatorname{sech}^2 x (3k^2 - 8 + 7ik \tanh x)].$$

Once again we choose  $u_1 = \psi_0$  and  $u_2 = e(x, k_0)$ . Up to an inessential constant factor the Wronskian of these functions reads

$$W(u_1, u_2) = e^{ik_0 x} w_0(x) \operatorname{sech}^7 x$$

where  $w_0(x) = A_1 \cosh 2x + A_2 \cosh 4x + B_1 \sinh 2x + B_2 \sinh 4x + c$  with  $A_1 = 4(16 + k_0^2)$ ,  $A_2 = 7k_0^2 - 8$ ,  $B_1 = -2ik_0(16 + k_0^2)$ ,  $B_2 = -ik_0(k_0^2 - 14)$  and  $c = -3(k_0^2 + 16)$ . Formula (18) yields the potential

$$V_2 = -\frac{6}{\cosh^2 x} + 2 \frac{[w_0'(x)]^2 - w_0''(x)w_0(x)}{w_0^2(x)}.$$

It has one discrete level  $E = -1$  and the spectral singularity  $E = k_0^2$ .

Now we discuss briefly the role of spectral singularities [2, 5, 11].

- (I) First of all we would like to stress that the spectral singularities play a very special role with respect to the other points of the continuous spectrum of the operator  $h$ . Hamiltonian  $h$  having spectral singularities has some new properties which appear neither in the theory of self-adjoint operators nor in the theory of non-Hermitian Hamiltonians with purely discrete spectrum and in particular in the theory of finite-dimensional Hamiltonians.
- (II) It happens that the whole Hilbert space  $L^2(R^+)$  can be presented as an orthogonal sum  $\mathcal{M} \dot{+} \mathcal{N}$  where  $\mathcal{M}$  corresponds to the continuous spectrum of  $h$  and  $\mathcal{N}$  is related to its discrete spectrum. Under condition (1) the space  $\mathcal{N}$  is finite-dimensional. If  $h$  has no spectral singularities being restricted to the space  $\mathcal{M}$  it is similar to a self-adjoint operator. This means that there exists a continuous in the  $L^2(R^+)$ -norm one-to-one mapping of the space  $\mathcal{M}$  onto itself which we denote by  $T$  such that  $T^{-1}hT$  is self-adjoint. If  $h$  has a spectral singularity such a representation is impossible.
- (III) With the operator  $h$  one can always associate the so-called 'resolution of the identity operator'  $P = P(\Delta)$  which is a set of projectors having properties similar to corresponding properties in the case of a Hermitian operator. It plays the role of a 'coordinate system' where the 'matrix' of the operator  $h$  takes the 'simplest' form. If  $h$  has no spectral singularities the resolution of the identity is bounded with respect to a norm. For  $h$  with spectral singularities the norm  $\|P(\Delta)\|$  tends to infinity when  $\Delta$  tends to a spectral singularity. In geometric language this may be interpreted as if the 'coordinate system' where  $h$  with a spectral singularity acquires the simplest form has such 'coordinate subspaces' that the angle between them may be as small as desired.
- (IV) According to a result by Levin [11] for any  $h$  without spectral singularities there exists for all  $f \in L^2(R^+)$  a series expansion over the set of eigenfunctions and associated functions converging in  $L^2(R^+)$ -norm. If  $h$  has a spectral singularity such an expansion exists not for all  $f \in L^2(R^+)$  but the set for which it exists is dense in  $L^2(R^+)$  [5]. This feature of a Hamiltonian with spectral singularities is due to the fact that for some values of  $k = k_j$  the Jost function vanishes, i.e. either  $A(k_j) = 0$  or  $A(-k_j) = 0$ . Just both  $A(k)$

and  $A(-k)$  appear in the denominator of the integral over the continuous spectrum in the Fourier series expansion of an element from the Hilbert space  $L^2(R^+)$  over the set of the eigenfunctions and associated functions of the Hamiltonian  $h$ . As a result at either  $k = k_j$  or  $k = -k_j$  the integrand has a pole and the integral becomes divergent. In [5] a regularization procedure for this integral is discussed in detail.

We leave open the question whether or not Hamiltonians with spectral singularities are acceptable in complex quantum mechanics. But we hope that in some cases SUSY transformations may be useful while working with such Hamiltonians.

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